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Translated by N.H.C.

PMM U.S.S.R.,Vol.47,No.6,pp.720-724,1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 62-50

ON A GAME OF OPTIMAL PURSUIT OF AN OBJECT BY TWO OTHERS*

A.G. PASHKOV and S.D. TEREKHOV

A game problem of the simple pursuit of one object by two others, the former having a speed advantage , is analyzed. The duration of the game is fixed. The payoff functional is the distance between the object being pursued and the pursuer closest to it when the game terminates. Similar problems were examined in /1—7/.

The motion of pursuers $P_i\,(y^{(i)})$ is described by the equations

$$y_1^{\cdot(i)} = u_1^{(i)},\ y_2^{\cdot(i)} = u_2^{(i)},\ |u^{(i)}| \leqslant \mu,\ \mu > 0 \tag{1}$$
$$(y^{(i)} = \{y_1^{(i)}, y_2^{(i)}\},\ u^{(i)} = \{u_1^{(i)}, u_2^{(i)}\},\quad i = 1, 2)$$

The object $E\,(z)$ being pursued moves in accordance with the equations

$$z_1^{\cdot} = v_1,\ z_2^{\cdot} = v_2;\ |v| \leqslant \nu,\ \nu > \mu\ \ (v = \{v_1, v_2\}). \tag{2}$$

Here $u^{(i)}$, v are the control vector. The time the game ends $t = \vartheta$ is fixed. The game's payoff γ is the distance between the object being pursued and the pursuer closest to it at the instant $t = \vartheta$, i.e.,

$$\gamma = \min\ (z_1(\vartheta) - y_1^{(i)}(\vartheta))^2 + (z_2(\vartheta) - y_2^{(i)}(v))^2]^{1/2} \tag{3}$$

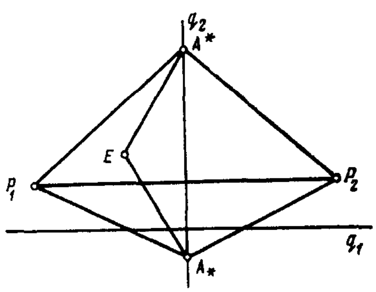

Fig.1

Henceforth we will assume that $|P_1^\circ P_2^\circ| > 0$. The case $P_1^\circ = P_2^\circ$ will be considered separately. In a plane we set up a fixed rectangular system of coordinates with axes q_1 and q_2. We direct the abscissa axis q_1 from the initial position of the first pursuer $P_1^\circ\,(y_0^{(1)})$ to the initial position of the second pursuer $P_2^\circ\,(y_0^{(2)})$. We direct the ordinate axis q_2 through the midpoint of the segment $[P_1^\circ P_2^\circ]$ perpendicular to it, so as to obtain a right-oriented system of coordinates (Fig.1). The domain of attainability $G^{(i)}\,(t, y^{(i)}, \vartheta)$ of the objects P_i $(i = 1, 2)$ from the position $\{t, y^{(i)}\,(t)\}$ by the instant $t = \vartheta$ is a circle of radius $r\,(t) = \mu\,(\vartheta - t)$ with centre at the point $\{y^{(i)}\,(t)\}$. The domain of attainability $G\,(t, z, \vartheta)$ of the object E from position $\{t, z\,(t)\}$ is a circle of radius $R\,(t) = \nu\,(\vartheta - t)$ with centre at the point $\{z\,(t)\}$. Suppose that at some instant t the object $P_i\ (i = 1, 2)$ is at the position $\{y_1^{(i)}\,(t),\ y_2^{(i)}\,(t)\}$, $y_1^{(1)}\,(t) = -y_1^{(2)}\,(t)$, $y_2^{(1)}\,(t) = y_2^{(2)}\,(t)$. At the instant t the object E is at position $\{z_1\,(t), z_2\,(t)\}$. The attainability domain $G\,(t,\ z\,(t),\ \vartheta)$ of the one being pursued intersects the axis q_2 at the points $A^*\,(0, q^*)$ and $A_*\,(0, q_*)$ (Fig.1)

$$q^* = z_2\,(t) + ((\nu\,(\vartheta - t))^2 - z_1^2\,(t))^{1/2} \tag{4}$$
$$q_* = z_2\,(t) - ((\nu\,(\vartheta - t))^2 - z_1^2\,(t))^{1/2}$$

We see that the distances between the pursuers P_i and the points A^*, A_* satisfy the following relations: $\operatorname{sign}\,(|P_iA^*| - |P_iA_*|) = \operatorname{sign}\,(z_2\,(t) - y_2^{(i)}\,(t))$.

It can be shown that the optimal programmed strategy for object E to evade pursuers P_i at the instant $t = \vartheta$ from a specified initial position $\{t_0, z_0\}$ will be the extremal control $v\,(t)\ (t_0 \leqslant t \leqslant \vartheta)$ directed towards the point A^* if

$$z_2\,(t_0) - y_2^{(i)}\,(t_0) > 0$$

*Prikl.Matem.Mekhan.,47,6,898-903,1983

and towards the point A_* if

$$z_2(t_0) - y_2^{(i)}(t_0) < 0$$

The positions for which

$$z_2(t) = y_2^{(i)}(t), \quad |z_1(t)| \leqslant |y_1^{(i)}(t)|$$

form a singular set S. The two extremal aiming points A^* and A_* correspond to points of the set S. In this case the optimal programmed evasion strategy of object E consists of the two extremal controls $v_1(t)$ and $v_2(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ directed towards the points A^* and A_*, respectively. Player E can choose any one of them.

First let us determine the programmed maximin γ_* for the differential game (1)—(3). In this case, when at the initial instant t_0 object E is inside the domain $Q(t_0)$ bounded by the segments $P_i^\circ A_*$, $P_i^\circ A^*$ $(i = 1, 2)$, i.e., when the inequalities

$$\begin{gathered}\nu(\vartheta - t_0) \geqslant |z_1(t_0)| \\ [(\nu(\vartheta - t_0))^2 - z_1^2(t_0)]^{1/2}(z_1(t_0))^{-1} \geqslant \\ (((\nu(\vartheta - t_0))^2 - (z_1(t_0))^2)^{1/2} + z_2(t_0) - y_2^{(i)}(t_0)) \, |y_1^{(i)}(t_0)|^{-1}\end{gathered} \tag{5}$$

are satisfied the programmed maximin has the form

$$\begin{gathered}\gamma_* = \max_{v(t)} \min_{u(t)} \gamma = \max(\gamma_1, \gamma_2); \quad \gamma_1 = \varphi(q^*), \quad \gamma_2 = \varphi(q_*) \\ \varphi(q) = ((y_2^{(i)}(t_0) - q)^2 + (y_1^{(i)}(t_0))^2)^{1/2} - \mu(\vartheta - t)\end{gathered} \tag{6}$$

If inequality (5) is not satisfied, object E lies outside the domain $Q(t_0)$. (In particular, this case holds if the attainability domain of object E does not intersect the q_2 axis.) Here the game (1)—(3) degenerates into a game with one evader and one pursuer, considered, for example, in /1/. In this case we find that the programmed maximin has the form

$$\gamma_* = ((z_1^\circ - y_1^{\circ(i)})^2 + (z_2^\circ - y_2^{\circ(i)})^2)^{1/2} + (\nu - \mu)(\vartheta - t_0) \tag{7}$$

Finally, if $|P_1P_2| = 0$, then

$$\begin{gathered}\gamma_* = ((z_1^\circ - y_1^\circ)^2 + (z_2^\circ - y_2^\circ)^2)^{1/2} + (\nu - \mu)(\vartheta - t_0) \\ (y_1^\circ = y_1^{\circ(1)} = y_1^{\circ(2)}, \; y_2^\circ = y_2^{\circ(1)} = y_2^{\circ(2)})\end{gathered} \tag{8}$$

It can be verified, for positions of player E for which the second inequality in (5) becomes an equality (i.e., E is located on one of the edges of the quadrangle $P_1A^*P_2A_*$), that functions (6) and (7) are equal together with all their derivatives. Thus, expressions (6) and (7) define a continuous function $\gamma_* \geqslant 0$ continuously differentiable for $t_0 \leqslant t < \vartheta$ everywhere except the set S.

Below we shall prove that the programmed maximin γ_* is identical with the value of differential game (1)—(3). When proving this fact we take advantage of the following circumstance. Consider a differential game in which (1) is replaced by the equation

$$\dot{y}_1^{(1)} = u_1, \quad \dot{y}_2^{(1)} = u_2, \quad \dot{y}_1^{(2)} = -u_1, \quad \dot{y}_2^{(2)} = u_2 \tag{9}$$

i.e., in (1) we set $u_1^{(1)} = -u_1^{(2)} = u_1$. If we compute the programmed maximin γ_{**} for the differential game (9), (2), (3), we obtain

$$\gamma_* = \gamma_{**} \tag{10}$$

Let ρ_* be the value of differential game (1)—(3) and ρ_{**} be the value of differential game (9), (2), (3). In game (9), (2), (3) there are additional constraints on the first pursuer. Therefore,

$$\rho_* \leqslant \rho_{**} \tag{11}$$

It is well known that a differential game's value and the programmed maximin are related by the inequality

$$\rho_* \geqslant \gamma_*, \quad \rho_{**} \geqslant \gamma_{**} \tag{12}$$

If the equation

$$\rho_{**} = \gamma_{**} \tag{13}$$

holds, then from (10)—(12) it follows that

$$\rho_* = \gamma_*$$

Let us prove (13). The equations

$$y_1^{(1)}(t) = -y_1^{(2)}(t), \quad y_2^{(1)}(t) = y_2^{(2)}(t)$$

always hold for system (9). The verification of (13) reduces to verifying the u-stability property /1/ for the function γ_{**}, i.e., for any position $\{t_*, y_*, z_*\}$ of game (9), (2), (3) and for any function $v(t)$ $(t_* \leqslant t \leqslant t^*)$ we can find a function $u(t)$ $(t_* \leqslant t \leqslant t^*)$ such that the pair of controls will take system (9), (2), (3) from position $\{t_*, y_*, z_*\}$ to position $\{t^*, y^*, z^*\}$

$$y^* = y(t^*), \quad z^* = z(t^*)$$

such that the inequality

$$\gamma_{**}(t^*, y^*, z^*) \leqslant \gamma_{**}(t_*, y_*, z_*) \tag{14}$$

is satisfied. The following cases are possible.

1) Suppose that at the initial instant player E is strictly inside the triangle $P_1A^*P_2$. Note that if the function γ_{**} is differentiable in a neighbourhood of the position being examined, then to prove (14) we can make use of the fundamental equation of differential game theory. In the case being examined we set $\gamma_{**} = \gamma_1$, where γ_1 is a differentiable function. We introduce the notation

$$r = ((\nu(\vartheta - t))^2 - z_1^2)^{1/2}, \quad q_1 = z_2 + r$$
$$R_1 = ((q_1 - y_2)^2 + y_1^2)^{1/2}, \quad \gamma_1 = R_1 - \mu(\vartheta - t)$$

We find the partial derivatives of the functions

$$\partial r/\partial t = -\nu^2(\vartheta - t)/r, \quad \partial r/\partial z_1 = z_1/r$$
$$\partial q_1/\partial t = -\nu^2(\vartheta - t)/r, \quad \partial q_1/\partial z_1 = -z_1/r, \quad \partial q_1/\partial z_2 = 1$$
$$\partial R_1/\partial y_1 = y_1/R_1, \quad \partial R_1/\partial y_2 = -(q_1 - y_2)/R_1$$
$$\partial R_1/\partial q_1 = (q_1 - y_2)/R_1, \quad \partial R_1/\partial z_1 = -(q_1 - y_2)z_1/(R_1 r)$$

$$\partial R_1/\partial z_2 = (q_1 - y_2)/R_1, \quad \partial\gamma_1/\partial t = \mu - \nu^2(\vartheta - t)(q_1 - y_2)/(R_1 r)$$
$$\partial\gamma_1/\partial y_1 = y_1/R_1, \quad \partial\gamma_1/\partial y_2 = -(q_1 - y_2)/R_1$$
$$\partial\gamma_1/\partial z_1 = -(q_1 - y_2)z_1/Rr, \quad \partial\gamma_1/\partial z_2 = (q_1 - y_2)/R_1$$

The fundamental equation has the form

$$\partial\gamma_1/\partial t + \min_u((\partial\gamma_1/\partial y_1)u_1 + (\partial\gamma_1/\partial y_2)u_2) + \max_v((\partial\gamma_1/\partial z_1)v_1 + (\partial\gamma_2/\partial z_2)v_2) = 0$$

Since $z_2 \geqslant y_2$, we have $q_1 - y_2 \geqslant 0$. Hence it follows that

$$\min_u((\partial\gamma_1/\partial y_1)u_1 + (\partial\gamma_1/\partial y_2)u_2) = -\mu$$
$$\max_v((\partial\gamma_1/\partial z_1)v_1 + (\partial\gamma_1/\partial z_2)v_2) = \nu^2(\vartheta - t)(q_1 - y_2)/(R_1 r)$$

Thus, the fundamental equation is satisfied. Similarly we can verify that the fundamental equation for positions of E located strictly inside the triangle $P_1A_*P_2$ is satisfied.

2) We consider the other case for game (9), (2), and (3). Let the positions of players P_i and E be such that either inequalities (5) are not satisfied or $P_1^\circ = P_2^\circ$. This signifies that either the evader E is outside the quadrangle $P_1A^*P_2A_*$ or the initial positions of the pursuers coincide. In both cases game (9), (2), (3) degenerates into a one-to-one game for which the validity of (14) is obvious.

3) Let us consider further the positions belonging to the singular set S defined above. We specify auxiliary controls of pursuers P_i by the following expressions:

$$u^{(i)}(t, y^{(i)}, z, v(t)) = \{(-1)^{i+1}(\mu^2 - (v_2(t))^2)^{1/2};\ v_2(t)\} \tag{15}$$

If $|v_2(t)| \leqslant \mu\,|q - y_2^{(i)}|\,/\,((y_1^{(i)})^2 + (q - y_2^{(i)})^2)^{1/2}$

$$u^{(i)}(t, y^{(i)}, z, v(t)) = \{-\mu y_1^{(i)}/((y_1^{(i)})^2 + (q - y_2^{(i)})^2)^{1/2};\ \mu(q - y_2^{(i)})/((y_1^{(i)})^2 + (q - y_2^{(i)})^2)^{1/2}\} \tag{16}$$

if

$$|v_2(t)| > \mu\,|q - y_2^{(i)}|\,/((y_1^{(i)})^2 + (q - y_2^{(i)})^2)^{1/2} \quad (i = 1, 2)$$

where q is the same as in (6).

3a) Suppose that during some sufficiently small interval Δt player E uses the control $v(t) = \{v_1(t),\ v_2(t)\}$, and $|v_1(t)| \leqslant \nu$, $v_2(t) = 0$. Then at the instant $t + \Delta t$ it is found at the point E' (Fig.2). In this case the pursuers P_i choose the extremal controls $u^{(i)}(t, y^{(i)}, z, v(t))$ described in (15) and directed along the vector $\mathbf{P}_i\mathbf{E}$. Players P_i move in accordance with these controls until one of them reaches player E. In the case when the coordinates of one of the P_i coincide with the evader's coordinates, game (9), (2), (3) degenerates into the two-player game considered in paragraph 2. If the evader chooses a control as shown above, he does not act optimally and the controls $u^{(i)}$ work against him in the sense that the value

γ_{**} is reduced .

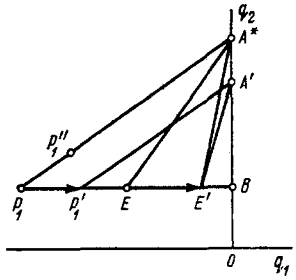

Fig.2

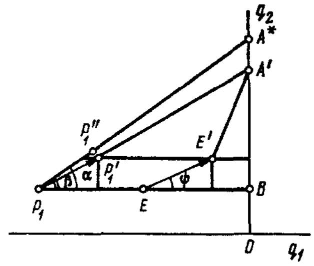

Fig.3

It is necessary to prove that (Fig.2)

$$|A^* P_1''| > |P_1'A'|$$

where P_1'' is the location of the pursuer P_1 at the instant $t + \Delta t$ in the case when E moves with maximum speed towards the point $A^* (0, q^*)$, while P_1 applies the control $u^{(1)}(t, y^{(1)}, z, v(t))$, corresponding to this case, having the form (15). We put

$$T = \nu - t,\ a = |y_1^{(1)}|,\ b = |z_1|,\ \bar{v} = v_1(t),\ |\bar{v}| \leqslant \nu$$

It is necessary to prove the inequality

$$((\nu T)^2 - b^2 + a^2)^{1/2} - \mu\Delta t > ((\nu(T - \Delta t))^2 - (b - \bar{v}\Delta t)^2 + (a - \mu\Delta t)^2)^{1/2}$$

It is sufficient to prove the inequality

$$\mu((\nu^2)^2 - b^2 + a^2)^{1/2} < \nu(\nu T - b) + a\mu$$

The truth of the latter inequality follows at once from the relations

$$\nu T - b > 0,\ \nu \geqslant \mu,\ ((\nu T)^2 - b^2 + a^2)^{1/2} < \nu T - b + a$$

Hence it follows that in the case being examined, as a result of the actions of the evader mentioned, the controls $u^{(i)}(t, y^{(i)}, z, v(t))$ prescribed by relation (15) furnish the pursuers P_i with a result better than γ_{**}.

3b) Now, during some sufficiently small interval Δt, let E use the control

$$v(t) = \{v_1(t), v_2(t)\},\ v_2(t) > 0$$

satisfying condition (15). Then, according to (15), player P_1 chooses the control

$$u(t) = \{u_1(t), u_2(t)\},\ u_2(t) = v_2(t)$$

After a time Δt player P_1 is in position P_1' (Fig.3). It is necessary to prove the inequality

$$|A^* P_1''| > |P_1'A'|$$

Put

$$a = |y_1^{(1)}|,\ b = |z_1|$$

and let ψ, α, β be the angles between the vectors EE', P_1A^*, P_1P_1', respectively, and the q_1 axis. The inequality to be proved takes the form

$$((\nu T)^2 - b^2 + a^2)^{1/2} - \mu\Delta t > ((\nu(T - \Delta t))^2 - (b - \bar{v}\Delta t\cos\psi)^2 + (a - \mu\Delta t\cos\beta)^2)^{1/2},\ |\bar{v}| \leqslant \nu \tag{17}$$

From the constraints on $v_2(t)$ in (15) it follows that

$$0 \leqslant \psi \leqslant \arcsin(\mu(\sin\alpha/\bar{v})),\ \mu\sin\alpha < \bar{v},$$
$$\mu\Delta t\sin\beta = \bar{v}\Delta t\sin\psi$$

It is sufficient to prove the inequality

$$((\nu T)^2 - b^2 + a^2)^{1/2} < (\nu/\mu)(\nu T - b\cos\psi + a((\mu/\nu)^2 - \sin^2\psi)^{1/2}) \tag{18}$$

The minimum of the right-hand side of (18) is reached when

$$\psi = \arcsin((\mu\sin\alpha)/\nu)$$

Thus, (17) is satisfied if the inequality

$$((\nu T)^2 - b^2 + a^2)^{1/2} < \nu T - b + a$$

holds. The truth of the latter inequality is easily established.

Thus, we have proved the u-stability property of the function γ_{**}. Hence, we have proved that the programmed maximin γ_* is identical with the value of differential game (1)–(3).

Note that all the constructions in the paper can be generalized to the case when $\mu \geqslant \nu$, i.e., the pursuers have the advantage in speed. The function γ_* can be investigated on the singular set S using the results obtained in /2/.

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PMM U.S.S.R.,Vol.47,No.6,pp. 724-727,1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 62-50

ON A DIFFERENTIAL ENCOUNTER GAME*

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A game of the encounter of two objects subject to viscous friction and control forces is examined. The sufficient conditions for the equality of the game's value to the programmed maximin are obtained under constraints of a general form.

A positional encounter game is described by differential equations with constraints on the admissible controls (U_j is a compactum)

$$x^{\cdot j} = y^j, \ y^{\cdot j} = -k_j y^j + u^j, \ j = 1, 2 \tag{1}$$
$$x^j, y^j, u^j \in E^n; \ u^j \in U_j, k_j \geqslant 0$$

by the termination time T and by a payoff functional minimizable by the first player and maximizable by the second

$$I(z^1(\cdot), z^2(\cdot)) = \| x^2(T) - x^1(T) \|, \ z^j = (x^j, y^j) \tag{2}$$

The formalization of the game is completed by the concepts and constructions in /1, 2/: position strategies, constructive motions, and game value.

Let $G_T = (-\infty, T] \times E^{4n}$, $\gamma_T(t_0, z_0^1, z_0^2)$ be the value of game (1), (2) from the initial position $(t_0, z_0^1, z_0^2) \in G_T$, $X_T^j(t_0, z_0^j)$ be the set of points $x^j = x^j(T)$ in E^n which all possible motions $z^j(\cdot), z^j(t_0) = z_0^j$, can hit at instant $t = T$. We introduce into consideration the quantity (the programmed maximin)

$$\varepsilon_T(t_0, z_0^1, z_0^2) = \max_{x^2 \in X_T^2(t_0, z_0^2)} \ \min_{x^1 \in X_T^1(t_0, z_0^1)} \| x^2 - x^1 \|$$

It is required to find the conditions under which

$$\gamma_T(t_0, z_0^1, z_0^2) = \varepsilon_T(t_0, z_0^1, z_0^2) \ \forall (t_0, z_0^1, z_0^2) \in G_T \tag{3}$$

In the isotropic case, i.e., when

$$U_j = \{u^j \in E^n \mid \| u^j \| \leqslant F_j\}, \ j = 1, 2 \tag{4}$$

a complete solution of the game is given in /3/; the sufficient conditions for (3) to be satisfied in this case have been given in /1, 2, 4/. We remark that the results mentioned do not carry over directly to the case of arbitrary U_j.

*Prikl.Matem.Mekhan.,47,6,904-908,1983